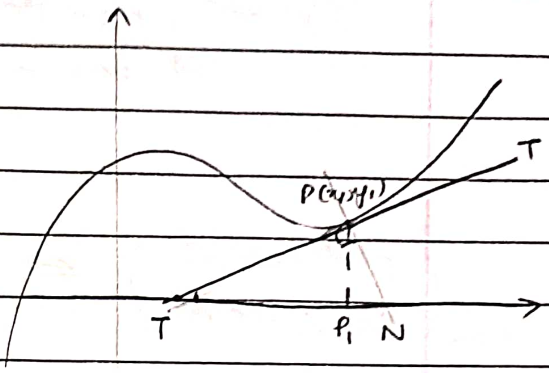
APPLICATION OF DERIVATIVES

07/06/2023

TANGENT & NORMALS

1. Eq^n of T at $P(x_1, y_1)$

$$(y-y_1) = \left(\frac{dy}{dx}\right)_{at(x_1,y_1)} (x-x_1)$$



2. Eq^n of N at $P(x_1, y_1)$

$$(y-y_1) = \frac{-1}{(dy/dx)_{at\ (x_1,y_1)}} (x-x_1)$$

3. If eq^n of the curve in parametric form

$x = f(t)$ $\quad$ $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}$

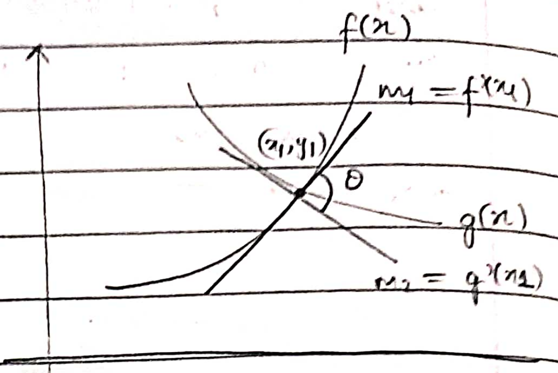
$y = g(t)$

4. Length of

- Tangent $\quad PT = |y| \sqrt{1+\left(\frac{dx}{dy}\right)^2}$
- Normal $\quad PN = |y| \sqrt{1+\left(\frac{dy}{dx}\right)^2}$
- Sub-tangent $\quad P_1T = \left|y\left(\frac{dx}{dy}\right)\right|$
- Sub-normal $\quad P_1N = \left|y\left(\frac{dy}{dx}\right)\right|$

→ Angle b/w 2 curves

$$t\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$



(orthogonal intersection) ⇔ $m_1 m_2 = -1$

Q. PT normal to the curve $x = a(c\theta + \theta s\theta)$, $y = a(s\theta - \theta c\theta)$ at any pt. θ is s.t it is at const. dist from origin

A. $$\frac{dy}{dx} = \frac{c\theta_1 - (c\theta_1 - \theta s\theta_1)}{-s\theta_1 + (s\theta_1 + \theta c\theta_1)} = t\theta_1$$

N: $m = \frac{-1}{t\theta_1} \Rightarrow x + t\theta_1 y = a(c\theta_1 + \theta_1 s\theta_1) + a t\theta_1 (s\theta_1 - \theta_1 c\theta_1)$

$$d_{(0,0)} = \frac{a(c\theta + \theta s\theta) + a\, t\theta\,(s\theta - \theta c\theta)}{\sqrt{1 + t\theta^2}}$$

$$= a(c^2 + \theta sc + s^2 - \theta sc)$$

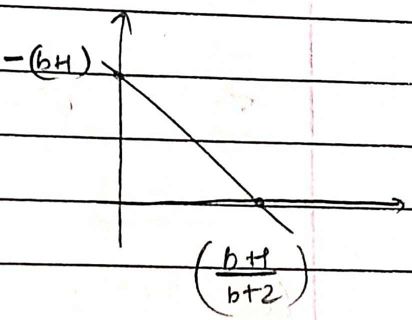
$$= \underline{a}$$

Q. The Δ formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the pt $(1,1)$ and the co-ordinate axes lies in the 1st Quadrant. If area of Δ is 2, find b.

A. $m = 2(1) + b = (b+2)$

T: $(b+2)x - y = (b+1) \Rightarrow \left(\frac{b+2}{b+1}\right)x + \left(\frac{-1}{b+1}\right)y = 1$

① $b+1 < 0 \Rightarrow \underline{b < -1}$



② $\frac{1}{2} \; -(b+1)\left(\frac{b+1}{b+2}\right) = 2$

$\Rightarrow b^2 + 2b + 1 = -4b - 8$

$\Rightarrow b^2 + 6b + 9 = 0 \Rightarrow \underline{b = -3}$

Q. Find the pts. on the curve $y^3 + 3x^2 = 12y$ where tangent is vertical.

A. $3y^2 y' + 6x = 12y' \Rightarrow y' = \frac{2x}{4-y^2}$

For vertical T, $y' = \infty \Rightarrow \underline{y = \pm 2}$

$\Rightarrow 3x^2 = 24 - 8, \; -24+8$

$= 16, \; -16$ ✗

$\Rightarrow \underline{x = \pm \frac{4}{\sqrt{3}}}$

Q. P.T tangent to the curve $y = e^x$ at the pt (c, e^c) intersects the line joining the pts $(c-1, e^{c-1})$ & $(c+1, e^{c+1})$ on the left of $x = c$.

A. L: $m = \dfrac{e^{(c+1)} - e^{(c-1)}}{c+1-c+1} = \dfrac{e^{(c+1)} - e^{(c-1)}}{2}$

$$\left(e^{(c+1)} - e^{(c-1)}\right)x - 2y = (c+1)e^{(c+1)} - (c+1)e^{(c-1)} - 2e^{(c+1)}$$
$$= (c-1)e^{(c+1)} - (c+1)e^{(c-1)}$$

T: $y' = e^x \Rightarrow e^c x - y = ce^c - e^c$
$$= (c-1)e^c$$

(Not required.)

$(L) - 2(T)$: $\left(e^{(c+1)} - 2e^c - e^{(c-1)}\right)x = (c-1)(e^{c+1} - 2e^c) - (c+1)e^{(c-1)}$
$$= (c-1)(e^{c+1} - 2e^c - e^{(c-1)}) - 2e^{(c-1)}$$

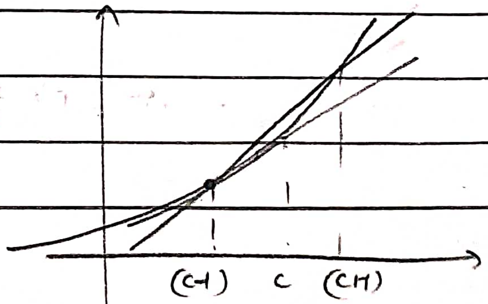
$$\Rightarrow x = c - 1 - \frac{2}{(e^2 - 2e - 1)}$$

$m_T = e^c \qquad m_L = \left(\dfrac{e^2-1}{2e}\right)e^c$

$\dfrac{m_L}{m_T} = \left(\dfrac{e^2-1}{2e}\right) > 1 \Rightarrow m_L > m_T$

$\Downarrow$

cuts on left.



Q. Find all the tangents to the curve $y = c(x+y)$, where $x \in [-2\pi, 2\pi]$, that are parallel to the line $\underline{x+2y=0}$

A. $y' = -s(x+y)\left(1+y'\right) \Rightarrow y' = \dfrac{-s(x+y)}{1+s(x+y)}$

ATQ $\quad y'_{(x,y)} = -\dfrac{1}{2} \Rightarrow \dfrac{1}{s(x+y)} + 1 = 2 \Rightarrow \underline{s(x+y)=1}$

$\Downarrow$

$c(x+y) = 0$

$\Rightarrow \boxed{y=0}$

$\Rightarrow \underline{cx = 0} \Rightarrow x = \underline{-\dfrac{3\pi}{2}, \underset{\times}{-\dfrac{\pi}{2}}, \dfrac{\pi}{2}, \underset{\times}{\dfrac{3\pi}{2}}}$

$\underline{sx \neq -1}$

$y' = -\dfrac{1}{2}$

① $x + 2y = \dfrac{\pi}{2}$ ② $x+2y = -\dfrac{3\pi}{2}$

Q. Find the eq^n of normal to the curve $y = (1+x)^y + s^{-1}(s^2x)$ at $x=0$.

A. N: $y = e^{y\,l(1+x)} + s^{-1}(s^2x)$

$$y' = (1+x)^y\left(\frac{y}{1+x} + y'\,l(1+x)\right) + \frac{1}{\sqrt{1-s^2x}}\,(2sx\,cx)$$

$x=0 \Rightarrow y = 1^y + s^{-1}(s^2 0) = 1$

$\Rightarrow y'_{(0,1)} = 1^1\left(\dfrac{1}{1+0} + y'\,l(1+0)\right) + 0 \Rightarrow \underline{y'=1}$

$m_N = -1 \Rightarrow x+y = 0+1$

$\Rightarrow x+y=1$

Q. The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2,0)$ and cuts the y-axis at a pt. Q where its gradient is 3. Find a, b, c.

A. 1. $0 = -8a + 4b - 2c + 5 \Rightarrow 8a - 4b + 2c = 5$

2. $x = 0 \Rightarrow y = 5$

A.T.Q $y'(0,5) = 3$.

$y' = 3ax^2 + 2bx + c$

$y'(0,5) = c \Rightarrow c = 3$

3. Since $f(x)$ just touches at $(-2,0)$

$y'(-2,0) = 0 \Rightarrow 12a - 4b + c = 0$

$\Rightarrow \left.\begin{matrix} 8a - 4b = -1 \\ \& \ 12a - 4b = -3 \end{matrix}\right\} \quad a = \frac{-1}{2} \quad b = \frac{-3}{4}$

$$(a, b, c) = \left(\frac{-1}{2}, \frac{-3}{4}, 3\right)$$

Q Find the cosine of the angle of intersection of the curves $y = 3^{(x-1)} l(x)$ & $y = x^x - 1$

A. Let (x_1, y_1) lie on both curves.

① $y' = \frac{3^{(x-1)}}{x} + 3^{(x-1)} l(x) l(3)$

② $y' = x^x (1 + l(x))$

Obviously, $x_1 = 1$ & $y_1 = 0$.

$\Rightarrow m_1 = 1 \quad m_2 = 1 \quad \Rightarrow \theta = t^{-1}(0) = 0$

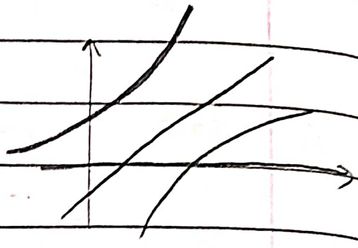
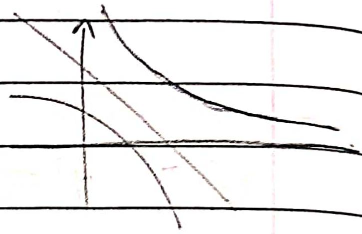
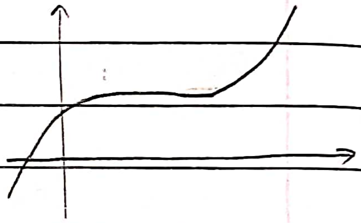
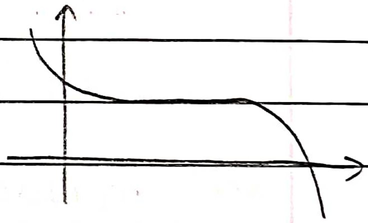
$\Rightarrow Co = 1$

MONOTONOCITY

Let $y = f(x)$ be a given fn with domain D.

Let $D_1 \subseteq D$, then

{ * Condⁿ: $f(x)$ cont on $[a,b]$ & diff in (a,b) }

Terms	Defⁿ $(\forall x_1, x_2 \in D)$	Basic Thm $(\forall x \in (a,b))$	Graphs
Increasing	$x_1 < x_2$ $\Rightarrow f(x_1) < f(x_2)$	$f'(x) > 0$	
Decreasing	$x_1 < x_2$ $\Rightarrow f(x_1) > f(x_2)$	$f'(x) < 0$	
Non-decreasing	$x_1 < x_2$ $\Rightarrow f(x_1) \leq f(x_2)$	$f'(x) \geq 0$	
Non-increasing	$x_1 < x_2$ $\Rightarrow f(x_1) \geq f(x_2)$	$f'(x) \leq 0$	

NOTE: If $f'(x) \geq 0 \;\; \forall \; x \in (a, b)$ & pts. which make $f'(x) = 0$ (in b/w (a, b)) don't form an interval, then $f(x)$ would be increasing in $[a, b]$

eg- $f(x) = x + Sx$

$f'(x) = 1 + Cx \geq 0$

but solⁿs of $f'(x) = 0$ do not form an interval.

Hence $x + Sx$ is increasing.

Q. Find the intervals of monotonocity of the following fn's.

1. $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 7$

2. $f(x) = -s^3_x + 3s^2_x + 5$, $x \in [-\pi/2, \pi/2]$

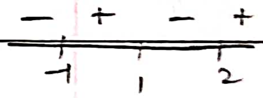
3. $f(x) = (2^x - 1)(2^x - 2)^2$

4. $f(x) = \frac{4s_x - 2x - xC_x}{2 + C_x}$; $x \in (0, 2\pi)$

5. $f(x) = t^{-1}(s_x + C_x)$; $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

A 1. $f'(x) = 12x^3 - 24x^2 - 12x + 24$

$= 12(x^3 - 2x^2 - x + 2) = 12(x-1)(x+1)(x-2)$



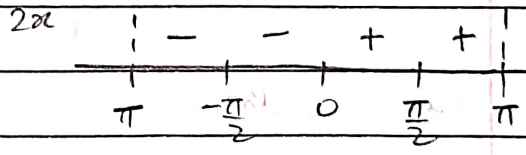
$\uparrow$: $[-1, 1] \cup [2, \infty)$

2. $f'(x) = -3s^2_x C_x + 6s_x C_x = 3s_x C_x(\underbrace{2 - s_x}_{>0})$

$= \frac{3(2 - s_x)}{2} \, s_{2x}$

$2x \in [-\pi, \pi]$

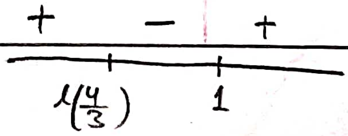
$\uparrow$: $[0, \pi/2]$



3. $f(x) = (k-1)(k^2 - 4k + 4) = (2^{3x} - 5 \cdot 2^{2x} + 8 \cdot 2^x - 4)$

$f'(x) = l(2)\,(3 \cdot 2^{2x} - 10 \cdot 2^{2x} + 8 \cdot 2^x)$

$= l(2)\,\underbrace{2^x}_{>0}\,(3 \cdot 2^x - 4)(2^x - 2)$

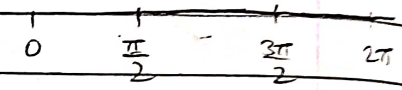


$\downarrow$: $[l(\frac{4}{3}), 1]$

4. $f(x) = \frac{4sx}{2+cx} - x$

$$f'(x) = 4\left[\frac{c(c+2)+s^2}{(c+2)^2}\right] - 1$$

$$= \frac{8c + \cancel{4} - c^2 - 4c - \cancel{4}}{(c+2)^2} = \frac{4c\overbrace{(4-c)}^{>0}}{(c+2)^2}$$



5. $f'(x) = \frac{c-s}{1+(c+s)^2} = \frac{\boxed{c-s}}{\underbrace{2+s2x}_{>0}} \Rightarrow tx < 1$

$\Rightarrow x \in \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

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Q1. P.T $f(x) = \frac{l(\pi+x)}{l(e+x)}$ ↓ in $(0, \infty)$

Q2. Let $f(x)$ & $g(x)$ be non ↓ & non ↑ from $[0, \infty) \longrightarrow [0, \infty)$

Let $h(x) = f(g(x))$
If $h(0) = 0$, then find $h(x) - h(1)$.

Q3. P.T $h(x) = f(x) - (f(x))^2 + (f(x))^3$ ↑ whenever $f(x)$ ↑ ∀ $x \in R$

Q4. If $f(x) = \frac{x}{\sin x}$ & $g(x) = \frac{x}{\tan x}$, $x \in (0,1]$

P.T $f(x) \uparrow$ & $g(x) \downarrow$

Q5. Find the interval of monotonocity of

(i) $f(x) = x^2 - l|x|$, $x \neq 0$

(ii) $f(x) = \frac{x}{l(x)}$

Q6. Let $f(x) = \begin{cases} x e^{ax}, & x \leq 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$, $a \in \mathbb{R}^+$

Find the interval where $f'(x) \uparrow$ & $\downarrow$

A. 1. $f'(x) = \frac{(\pi+x)\, l(e+x) - (e+x)\, l(\pi+x)}{l(e+x)^2}$

N: $(\pi+x)(e+x)\left[\frac{l(e+x)}{(e+x)} - \frac{l(\pi+x)}{(\pi+x)}\right]$

$g(x) = \frac{l(x)}{x} \Rightarrow g'(x) = \frac{1 - l(x)}{x^2}$

if $x > e \Rightarrow l(x) > 1 \Rightarrow 1 - l(x) < 0$

$\Rightarrow g'(x) < 0 \Rightarrow g(x) \downarrow$

$\Rightarrow \frac{l(x+\pi)}{(x+\pi)} < \frac{l(e+x)}{(e+x)} \Rightarrow f'(x) > 0 \Rightarrow f(x) \uparrow$

2. $x_1 < x_2 \Rightarrow g(x_1) \geq g(x_2)$

$\Rightarrow f(g(x_1)) \geq f(g(x_2))$

$\Rightarrow h(x_1) \geq h(x_2)$

$\forall \quad x > 0 \Rightarrow h(x) \leq h(0) = 0.$

$\Rightarrow h(x) = 0 \quad \forall \quad x \in \mathbb{R}.$

$\Rightarrow \underline{h(x) - h(1) = 0}$

3. $$h'(x) = f'(x) \left[1 - \underbrace{2f(x) + 3f(x)^2}_{D<0 \Rightarrow \quad >0 \ \forall x \in \mathbb{R}} \right]$$

$\Rightarrow f'(x) \uparrow \Rightarrow h'(x) \uparrow$

4. $$f'(x) = \frac{s_x - x c_x}{s_x^2} \qquad \because \quad x \in (0,1]$$

$\Rightarrow x < t_x$

$>0 \qquad \Rightarrow \underline{s - xc > 0}$

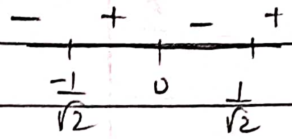
$$g'(x) = \frac{t_x - x\sec^2(x)}{t_x^2} = \frac{sc - x}{s^2} = \frac{s2x - 2x}{2s^2}$$

<0

$\because \quad x \in (0,1].$

$2x > s2x$

5. (i) $f'(x) = 2x - \frac{1}{x} = \frac{(\sqrt{2}x-1)(\sqrt{2}x+1)}{x}$



$f(x) \downarrow$, $\underline{x \in \left(-\infty, \frac{-1}{\sqrt{2}}\right) \cup \left(0, \frac{1}{\sqrt{2}}\right)}$

(ii) $f'(x) = \frac{\ell(x) - 1}{\ell^2(x)} \Rightarrow f(x) \downarrow$, $\underline{x \in (0, 1)}$

6. $$f'(x) = \begin{cases} (1+ax)e^{ax}, & x \leq 0 \\ 1 + 2ax - 3x^2, & x > 0 \end{cases}$$

$f''(x) = (2a + a^2x)e^{ax} = a^2\left(x + \frac{2}{a}\right)\underbrace{e^{ax}}_{>0} \quad \forall x \in \mathbb{R}$

$x > \frac{-2}{a} \Rightarrow f''(x) > 0 \Rightarrow f'(x) \uparrow \quad \forall x \in \underline{\left(\frac{-2}{a}, \infty\right)}$

Q. P.T $2sx + tx \geqslant 3x$, $x \in [0, \pi/2)$

A. $f(x) = 2sx + tx - 3x$

$$f'(x) = 2cx + \sec^2(x) - 3$$

$$= cx + cx + \frac{1}{c_x^2} - 3 \geqslant \overset{(AM \geqslant GM)}{\overset{\downarrow}{3}} - 3 = 0$$

$\Rightarrow$ $f(x)$ non-$\downarrow$, $\Rightarrow$ $f(x) \geqslant f(0)$

given $x \geqslant 0$ $\Rightarrow$ $2sx + tx \geqslant 3x$

Q1 P.T

(i) $\frac{x}{1+x} \leq l(1+x) \leq x$, $\forall\, x \geqslant 0$

(ii) $x - \frac{x^3}{6} \leq sx \leq x$, $x \in [0, \pi/2]$

(iii) $1 + x\ln(x + \sqrt{x^2+1}) \geqslant \sqrt{1+x^2}$, $\forall\, x \geqslant 0$

Q2 Use $f(x) = x^{1/x}$, $x > 0$ to determine the larger of e^π & π^e

Q3. Let $g(x) = 2f(\frac{x}{2}) + f(2-x)$ & $f''(x) < 0$ $\forall\, x \in (0,2)$
find the interval of ↑ & ↓ of $g(x)$

A1 (i) $f(x) = x - \ell(1+x) \Rightarrow f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$

$\because x \ge 0 \Rightarrow f(x) \ge f(0) \Rightarrow f(x) \uparrow$

$\Rightarrow x - \ell(1+x) \ge 0$

$\Rightarrow \underline{\ell(1+x) \le x}$

$g(x) = \ell(1+x) - \frac{x}{1+x} \Rightarrow g'(x) = \frac{1}{(1+x)} - \left[\frac{(1+x) - x}{(1+x)^2}\right]$

$= \frac{x}{(1+x)^2} > 0$

$\because x \ge 0 \Rightarrow g(x) \ge g(0)$

$\Rightarrow \underline{\ell(1+x) \ge \frac{x}{1+x}} \Rightarrow f(x) \uparrow$

(ii) $g(x) = sx - x + \frac{x^3}{6} \Rightarrow g'(x) = c - 1 + \frac{x^2}{2}$

$= \frac{x^2}{2} - 2s^2\left(\frac{x}{2}\right)$

$\forall \; x \in [0, \frac{\pi}{2}]$

$s(x/2) \le \frac{x}{2} \Rightarrow 2s^2\left(\frac{x}{2}\right) \le \frac{x^2}{2} \Rightarrow \frac{x^2}{2} - 2s^2\left(\frac{x}{2}\right) \ge 0$

$\Rightarrow s^2\left(\frac{x}{2}\right) \le \frac{x^2}{4}$

$\Rightarrow g'(x) \ge 0$

$\Rightarrow g(x) \uparrow$

Given $x \ge 0 \Rightarrow g(x) \ge g(0) \Rightarrow sx \ge x - \frac{x^3}{6}$

(iii) $f(x) = 1 + x\,l(x+\sqrt{x^2+1}) - \sqrt{1+x^2}$

$$f'(x) = l(x+\sqrt{x^2+1}) + \frac{x}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) - \frac{x}{\sqrt{x^2+1}}$$

$$= l(\underbrace{x+\sqrt{x^2+1}}_{\geq 1})$$

≥ 0

$\Rightarrow f(x) \uparrow$ Given $x \geq 0$

$\Rightarrow f(x) \geq f(0)$

$\Rightarrow 1 + x\,l(x+\sqrt{x^2+1}) \geq \sqrt{1+x^2}$

A2 $f(x) = e^{\frac{l(x)}{x}}$

$$f'(x) = x^{1/x}\left(\frac{\frac{1}{x^2} - l(x)}{x^2}\right) = x^{1/x}\left(\frac{1-x^2 l(x)}{x^4}\right)$$

$\forall\ x \geq e \Rightarrow l(x) \geq 1 \Rightarrow x^2 l(x) \geq e^2$

$\Rightarrow f'(x) < 0 \Rightarrow f(x) \downarrow$

$\pi > e \Rightarrow f(\pi) < f(e)$ & since $e^\pi > e^e$

$\Rightarrow e^e > \pi^\pi$ & $\pi^\pi > \pi^e$

$\Rightarrow e^\pi > \pi^e$

A3. $f''(x) < 0 \Rightarrow f'(x) \downarrow$

$g'(x) = f'\left(\frac{x}{2}\right) - f'\left(2-x\right)$

if $\frac{x}{2} < 2-x \Rightarrow x < 4-2x \Rightarrow \underline{x < \frac{4}{3}}$

$\Rightarrow f'\left(\frac{x}{2}\right) > f'\left(2-x\right) \Rightarrow g'(x) > 0 \Rightarrow g(x) \uparrow$

$\Rightarrow \underline{x \in \left(0, \frac{4}{3}\right)}$

Q. Let $a+b=4$, where $a<2$ & let $g(x)$ be a diff fn^n.
If $g'(x) > 0$, $\forall x$., P.T

$$\int_0^a g(x)\,dx + \int_0^b g(x)\,dx \uparrow \quad \text{as} \quad (b-a) \uparrow$$

A. $a = 2-t \qquad b = 2+t \qquad t \in (0, 2]$

$$h(t) = f(x) = \int_0^{(2-t)} g(x)\,dx + \int_0^{(2+t)} g(x)\,dx$$

$$h'(t) = \frac{d}{dt} f(x) = g(a)(-1) + g(b)(1)$$
$$= g(2+t) - g(2-t)$$

$$h''(t) = \frac{d^2}{dt^2} f(x) = \underbrace{g'(2+t)}_{>0} + \underbrace{g'(2-t)}_{>0} \Rightarrow h''(t) > 0$$
$$\Rightarrow h'(t) \uparrow$$

$\Rightarrow \quad h'(t) > h'(0) \quad \Rightarrow h'(t) > 0$

$\Rightarrow h(t) \uparrow$

$\because \quad t > 0 \Rightarrow 2t > 0 \Rightarrow (2+t) - (2-t) > 0$

$\Rightarrow (b-a) > 0$

$\Rightarrow \quad (b-a) \uparrow \quad \Rightarrow \quad h(t) \uparrow$

MAXIMA & MINIMA

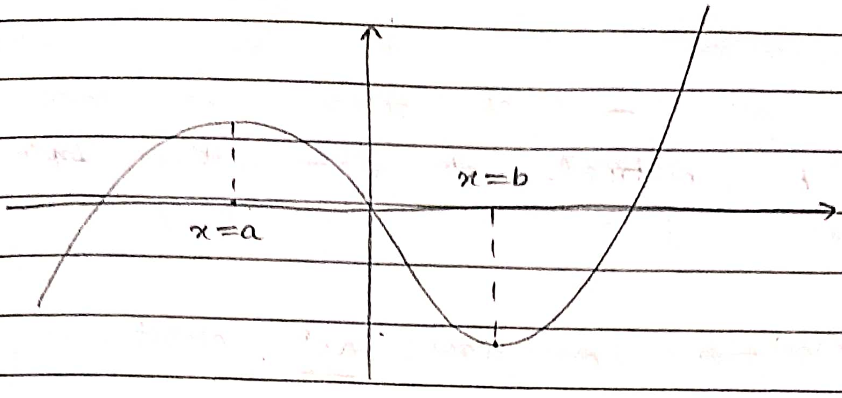
A fn^n $y = f(x)$ is said to have a local max. at $x = a$ if $f(a)$ is greatest of all values in the suitably small nbd of $x = a$, where $x = a$ is an interior pt. in the domain of $y = f(x)$.

i.e $\quad f(a) \geq f(a+h)$, $h \rightarrow$ finitely
& $\quad f(a) \geq f(a-h)$ small (+ve) no.

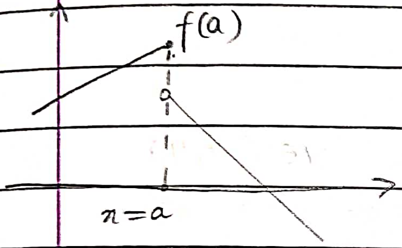
Similary, $x = b$ is local min.

if $\quad f(b) \leq f(b+h)$, $h \rightarrow$ finitely
$f(b) \leq f(b-h)$ small (+ve) no.

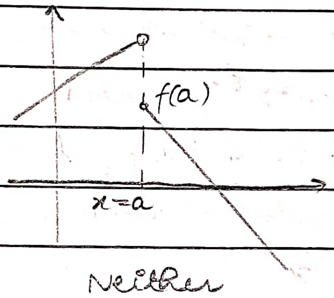
NOTE: End pts. of an interval can also be local max/min.



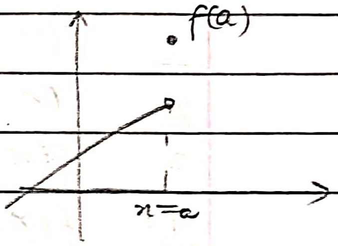
→ Max. & Min of non-diff (discont.) $f x^n$



Max



Neither



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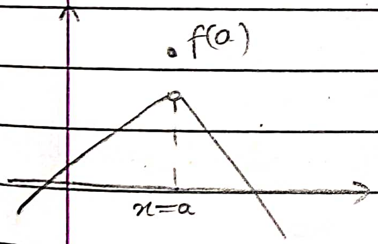

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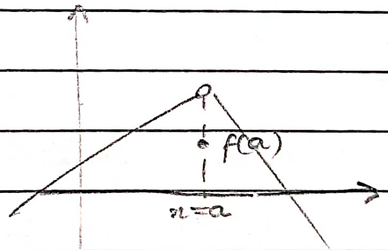
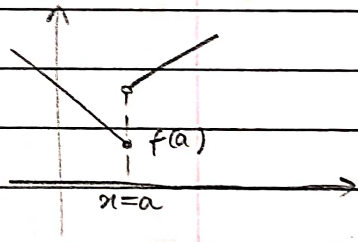


Min



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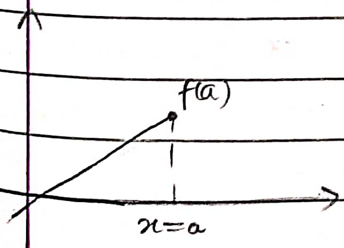

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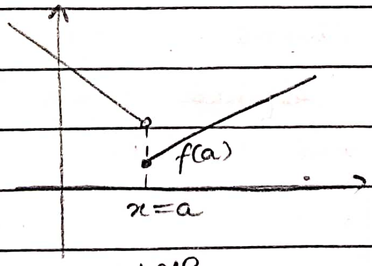

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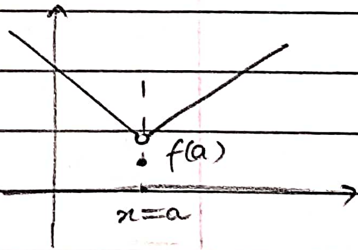
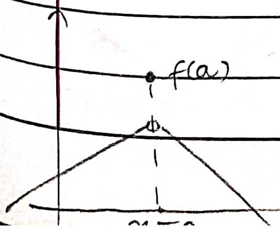
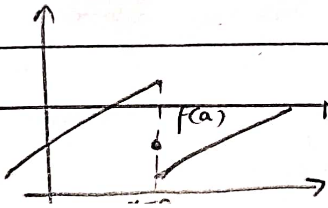


Min



Max



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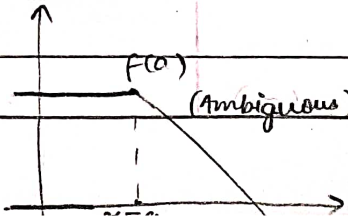


(Ambiguous)

07/06/2023

- **Critical pt.** – pt $x=a$ is said to be a critical pt. of the fn $y=f(x)$ if:

i) $f'(a)=0$ or does not exist
ii) $a \in D(f)$

Q Find the critical pts. of

1. $f(x) = x^3 - 3x^2 - 9x + 20$
2. $f(x) = (x-2)^{2/3}(2x+1)$
3. $f(x) = \min(\{x\}, \{-x\})$; $x \in (-4, 4)$
4. $f(x) = \dfrac{|2-x|}{x^2}$

A. 1. $f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1) = 0$

$\underline{x = -1, 3}$

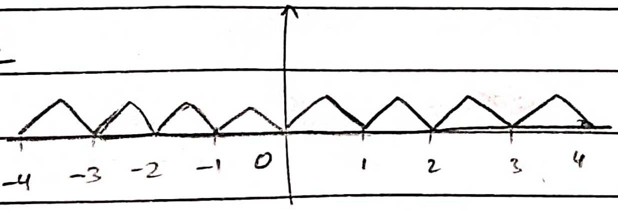
2. $f'(x) = \dfrac{2}{3}\dfrac{(2x+1)}{(x-2)^{1/3}} + 2(x-2)^{2/3} = 0$

$\Rightarrow \dfrac{(2x+1)}{3} = -(x-2) \Rightarrow 2x+1 = 6-3x$

$\Rightarrow \underline{x = 1}$

$f'(2)$ not defined $\Rightarrow x=2 \Rightarrow \underline{x = 1, 2}$

3.



$x = n + \frac{1}{2}$, $n \in \mathbb{Z}$

& n

4. $$f(x) = \begin{cases} \dfrac{x-2}{x^2}, & x \geq 2 \\ \dfrac{2-x}{x^2}, & x < 2, \quad x \neq 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} \dfrac{x^2 - 2x(x-2)}{x^4} = \dfrac{4-x}{x^3}, & x \geq 2 \\ \dfrac{-x^2 + 2x(x-2)}{x^4} = \dfrac{x-4}{x^3}, & x < 2 \end{cases}$$

$f'(2)$ does not exist $\Rightarrow$ $\underline{x=2}$

$f'(x) = 0 \Rightarrow x-4=0 \Rightarrow \underline{x=4}$

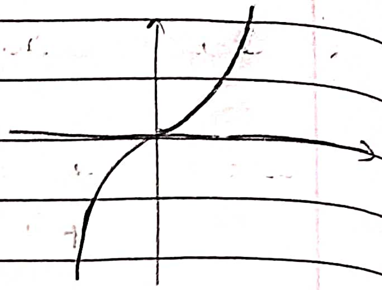
→ Methods for finding local extrema

Theorem: If $f(x)$ has local extremum at $x=c$, then either $f'(c)=0$ or $f'(c)$ does not exist.

The converse of this thm is not always true i.e. $f'(c)=0$ or does not exist does not necessarily imply that $f(x)$ has local extremum at $x=c$.

eg – $f(x) = x^3$

$f'(x) = 3x^2 = 0 \Rightarrow \underline{x=0}$



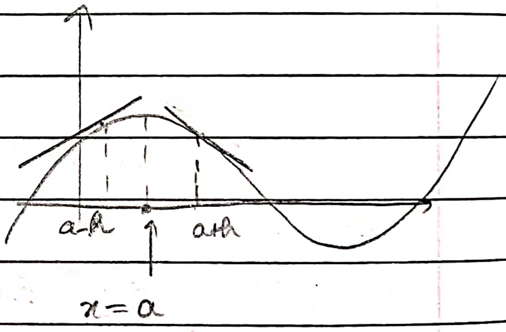
But $x=0$ is not a local extremum.

① First Derivative Test

Let $y = f(x)$ be cont & diff in nbd(c)

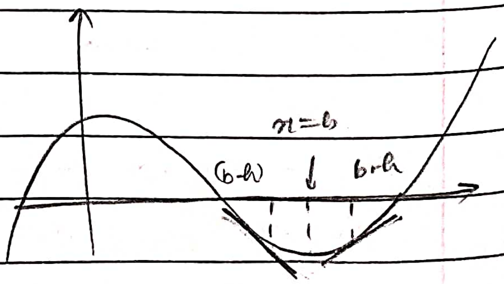
Pt $x = a$ is local max if

$f'(a+h) < 0$
$f'(a-h) > 0$



Pt. $x=a$ is local min if

$f'(b+h) > 0$
$f'(b-h) < 0$



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② Second Derivative Test

Let $y = f(x)$ be cont. & twice diff in nbd(c)
& $f'(c) = 0$

$f(x)$ has local max at $x=c$ if $f''(c) < 0$
$f(x)$ has local min at $x=c$ if $f''(c) > 0$

NOTE: When $f''(c) = 0$, the 2nd derivative test fails. The pt could be local max, local min or pt. of infliction.

③ nth Derivative Test.

Let $y = f(x)$ be a fn s.t.

i) $f'(c) = f''(c) = \dots = f^{(n-1)}(c) = 0$
ii) $f^{n}(c) \neq 0$

① if $n \in$ Even

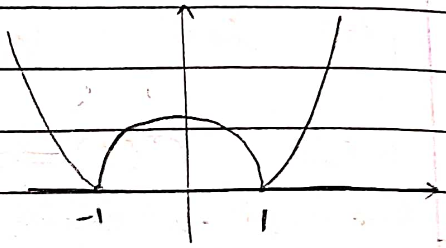
1.1 $f(x)$ has local max at $x=c$
if $f^{n}(c) < 0$

1.2 $f(x)$ has local min at $x=c$
if $f^{n}(c) > 0$

② if $n \in$ Odd
$f(x)$ has no local extremum at $x=c$

- Pt. of inflection — A pt. where the graph of fxn is cont. & has a tangent line & where the concavity changes is called pt. of inf.

eg $f(x) = |x^2-1|$



Here, on -1 & 1 concavity changes but tangent does not exist.

So, f(x) has no pt. of inf.

Q Investigate the ‸existence of local extrema of the following fxn at the given pts.

1. $f(x) = \{x\}$, $x=2$
2. $f(x) = \begin{cases} |x| , & x \neq 0 \\ 0 , & x=0 \end{cases}$, $x=0$
3. $f(x) = \begin{cases} x^3+x^2-10x , & x<0 \\ 3sx , & x \geq 0 \end{cases}$, $x=0$
4. $f(x) = \begin{cases} s\left(\frac{\pi x}{2}\right) , & x<1 \\ 3-2x , & x \geq 1 \end{cases}$, $x=1$.

DATE
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A. 1. f not cont. at $x=2$. So FDT cannot be applied. By dif $f(2) < f(2-h)$ ⇒ L. Min
$f(2) < f(2+h)$

2. $\left.\begin{array}{l} f'(0+h) = 1 \\ f'(0-h) = -1 \end{array}\right\}$ L. Min

3. $f'(x) = \begin{cases} 3x^2 + 2x - 10 , & x < 0 \\ 3cx , & x \geq 0 \end{cases}$ $\left.\begin{array}{l} f'(0+h) > 0 \\ f'(0-h) < 0 \end{array}\right\}$ L. Min

4. $f'(x) = \begin{cases} \frac{\pi}{2} c_{\frac{\pi}{2}x} , & x < 1 \\ -2 , & x \geq 1 \end{cases}$

$\left.\begin{array}{l} f'(1+h) = -2 \\ f'(1-h) > 0 \end{array}\right\}$ ⇒ L. Max

Q. Let $f(x) = \begin{cases} -x^3 + \left(\frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}\right) , & x \in [0,1) \\ 2x - 3 , & x \in [1,3] \end{cases}$

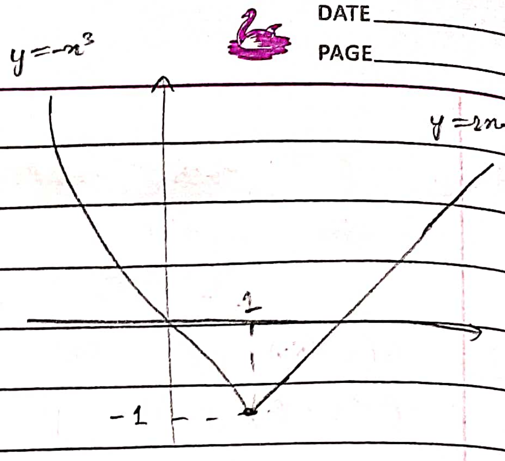
Find all possible real values of 'b' s.t $f(x)$ has the smallest value at $x=1$.

A. $f(1) = -1$ ⇒ $\forall x \in [0,1)$ $f(x) \geq -1$

So, $\lim_{x \to 1} f(x) = \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} - 1 \geq -1 \Rightarrow \frac{(b^2+1)(b-1)}{(b+1)(b+2)} \geq 0$

$b \in (-2, -1) \cup [1, \infty)$

Graphical Analysis -



so, $\dfrac{b^3-b^2+b-1}{b^2+3b+2} \geq 0$

Q1. Investigate the pts. of local extrema of $f(x) = \int_1^x 2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2 \, dt$

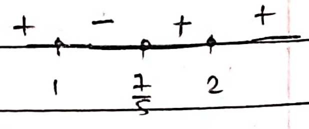
Q2. Find the pts. of infliction of

1. $f(x) = \sin x$
2. $f(x) = 3x^4 - 4x^3$
3. $f(x) = x^{1/3}$.

A. 1. $f'(x) = 2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2$

$= (x-1)(x-2)\left[2x^2 - 8x + 8 + 3x^2 - 9x + 6\right]$

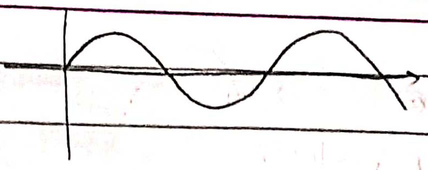
$= (x-1)(x-2)^2(5x-7)$



$\left.\begin{array}{l} f'(1+h) < 0 \\ f'(1-h) > 0 \end{array}\right\}$ L Max

$\left.\begin{array}{l} f'\left(\frac{7}{5}+h\right) > 0 \\ f'\left(\frac{7}{5}-h\right) < 0 \end{array}\right\}$ L Min

2.1 $x = n\pi$



2.2 $f'(x) = 12x^3 - 12x^2$

$f''(x) = 12(3x^2 - 2x) = 12x(3x-2) \Rightarrow x = 0, 2/3$

$+ \quad - \quad +$

$0 \quad 2/3$

2.3 $f'(x) = \dfrac{1}{3x^{2/3}}$ $\quad f''(x) = \dfrac{2}{9} x^{-5/3}$

$- \quad +$

0

$\Rightarrow x = 0$

- Global Maxima & Minima –

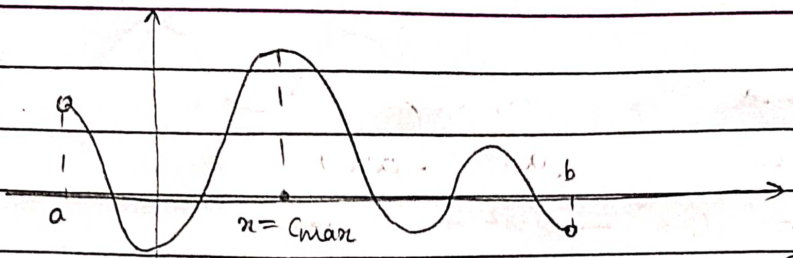
(I) Closed interval $[a, b]$

– Calc. C.P.s of $y = f(x)$.

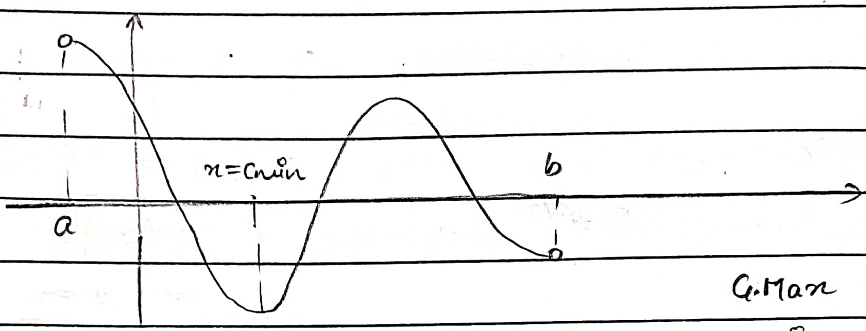
G. Max $= x$ of $\max\left(f(a), f(b), f(c_1) \ldots, f(c_n)\right)$

G. Min $= x$ of $\min\left(f(a), f(b), f(c_1) \ldots, f(c_n)\right)$

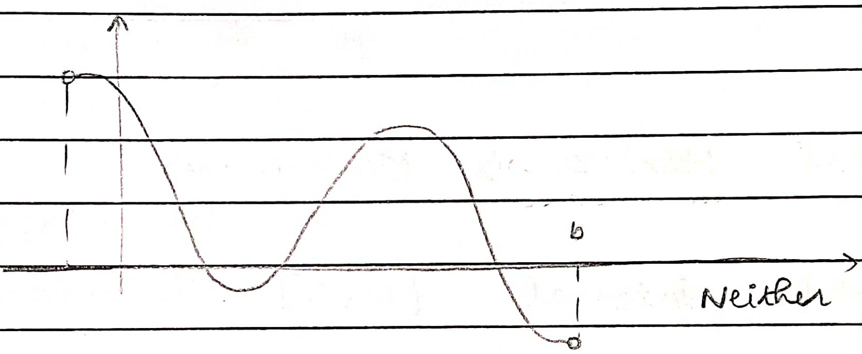
(II) Open interval (a, b)



G.Max exists
G.Min does not exist



G.Max does not exist
G.Min exists



Neither exist

Q. Let $f(x) = 2x^3 - 9x^2 + 12x + 6$
Discuss global max & min of $f(x)$ in
(i) $[0, 2]$
(ii) $(1, 3)$

A. $f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2)$

CP: $x = 1, 2$

(i)
$f(0) = 6$
$f(1) = 2 - 9 + 12 + 6 = 11$
$f(2) = 16 - 36 + 24 + 6 = 10$

G.Max : $x = 1$
G.Min : $x = 0$

(ii)
$f(1) = 11$
$f(2) = 10$
$f(3) = 54 - 81 + 36 + 6$
$= 15$

G.Max : Does not exist
G.Min : $x = 2$

Q. P.T $\frac{d^2x}{dy^2} = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

A. $\frac{dy}{dx} = \frac{1}{(dx/dy)} \Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{(dx/dy)^2}\frac{d\,(dx/dy)}{dx}$

$$= -\left(\frac{dy}{dx}\right)^2 \frac{d}{dy}\left(\frac{dx}{dy}\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2x}{dy^2} = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

08/06/2023

→ Analysis of cubic polynomial

Let $f(x) = x^3 + bx^2 + cx + d$

$f'(x) = 3x^2 + 2bx + c$

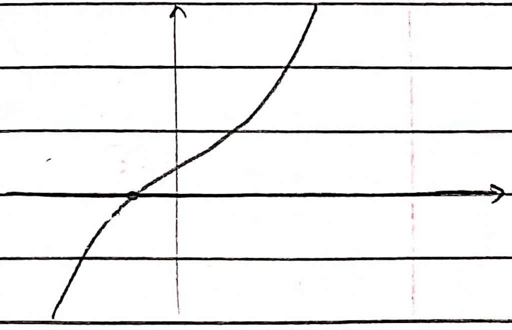
$D(f'(x)) = 4b^2 - 12c$

Case (I) : If $D<0 \Rightarrow f'(x) > 0 \;\forall\, x \in \mathbb{R}$

$\Rightarrow f(x) \uparrow \;\forall\, x \in \mathbb{R}$

$f(x)$ cuts x-axis only once

So, $f(x)=0$ has exactly one real rt.



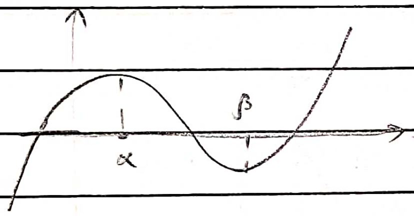
Possible graph

Case (II): If $D>0$, $\Rightarrow$ $f'(x)$ has 2 distinct real rts (say α & β)

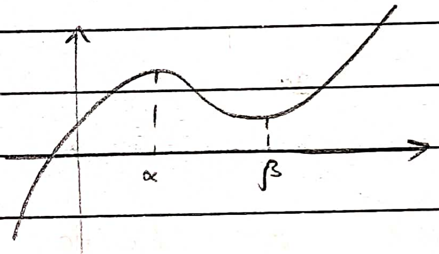
$$f'(x) = 3(x-\alpha)(x-\beta)$$

$\Rightarrow$ $f(x)\uparrow$, $x \in (-\infty, \alpha) \cup (\beta, \infty)$
$f(x)\downarrow$, $x \in (\alpha, \beta)$

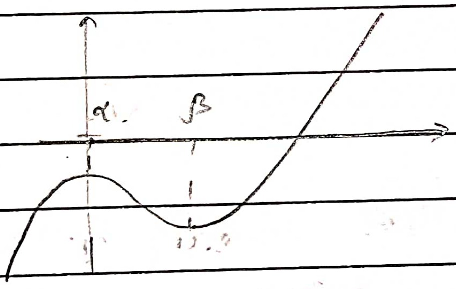
Possible graphs



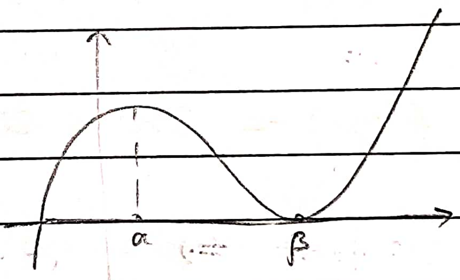
$f(\alpha)>0$, $f(\beta)<0$



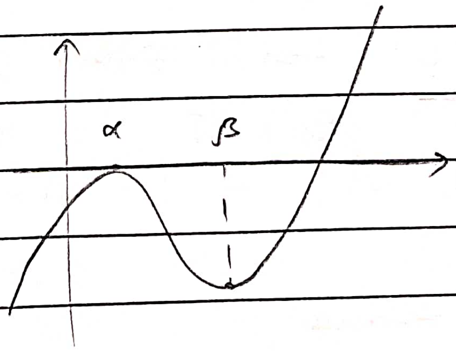
$f(\alpha)>0$, $f(\beta)>0$



$f(\alpha)<0$, $f(\beta)<0$



$f(\alpha)>0$, $f(\beta)=0$



$f(\alpha)=0$, $f(\beta)<0$

So, 1) $f(\alpha) f(\beta) > 0 \Rightarrow f(x) = 0$ has exactly one real rt.

2) $f(\alpha) f(\beta) < 0 \Rightarrow f(x) = 0$ has 3 distint real rts

3) $f(\alpha) f(\beta) = 0 \Rightarrow f(x) = 0$ has 3 real rts, one repeated

Case (II) : If $D = 0 \Rightarrow f'(x) = 3(x-\gamma)^2$

$$\Rightarrow f(x) = (x-\gamma)^3 + \lambda$$

If $\lambda = 0 \Rightarrow f(x) = 0$ has 3 equal real rts

$\lambda \neq 0 \Rightarrow f(x) = 0$ has exactly one real rt.

Q1: Find the value of 'a' if $x^3 - 3x + a = 0$ has 3 distinct real rts.

Q2 For what value of 'a' does the fxn $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 2$ have a +ve pt. of L.Max.

A. **1.** $f'(x) = 3x^2 - 3 = 0 \Rightarrow x = 1, -1$

A.T.Q $f(1)\,f(-1) < 0 \Rightarrow (a-2)(a+2) < 0$

$\Rightarrow \underline{a \in (-2, 2)}$

2. $f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9) = 0 \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$

$(\alpha \le \beta)$

A.T.Q $\alpha > 0 \Rightarrow \underline{\alpha, \beta > 0}$

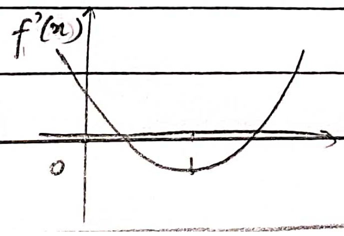
① $\underline{D > 0} \Rightarrow 36(a-7)^2 > 4(9)(a^2-9)$

$\Rightarrow a^2 - 14a + 49 > a^2 - 9$

$\Rightarrow 14a < 58 \Rightarrow \underline{a < \frac{29}{7}}$

② $\frac{-b}{2a} > 0 \Rightarrow 7 - a > 0 \Rightarrow \underline{a < 7}$

③ $\underline{f'(0) > 0} \Rightarrow (a^2-9) > 0 \Rightarrow \underline{a \in (-\infty, -3) \cup (3, \infty)}$



$\Rightarrow \underline{a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)}$

13/06/2023

MEAN VALUE THEOREM

→ Rolle's Theorem

Let $y = f(x)$ be a fxn satisfying.

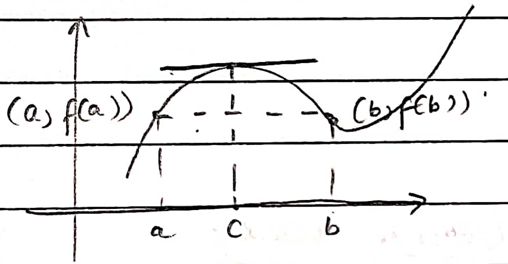
① f is cont. in $[a, b]$

② f is diff in (a, b)

③ $f(a) = f(b)$

Then, $\exists\ c \in (a, b)$ s.t $f'(c)=0$

Geometrically,



Analytically, in nbd $(x=c)$

$$f(c+h) - f(c) \leq 0 \quad \& \quad f(c-h) - f(c) \leq 0$$

$$\Rightarrow \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0 \quad \& \quad \lim_{h\to 0^+} \frac{f(c-h)-f(c)}{-h} \geq 0$$

$$\Rightarrow f'(c^+) \leq 0 \qquad \Rightarrow f'(c^-) \geq 0$$

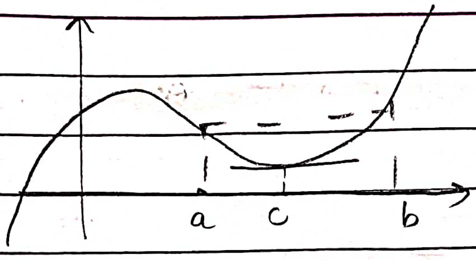
Since f is diff in $x \in (a, b)$

$$\Rightarrow f'(c^+) = f'(c^-) = 0$$

$$\Rightarrow \underline{f'(c) = 0}$$

□

Similarly.



→ Lagrange's Mean Value Theorem (LMVT)

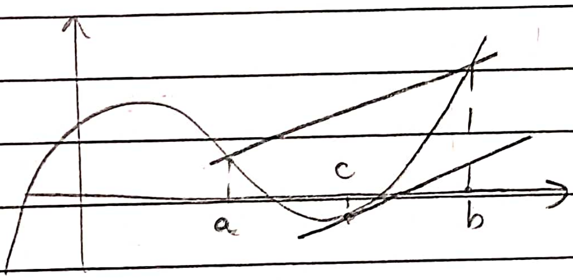
Let $y = f(x)$ be a fn satisfying

① f is cont. in $[a, b]$

② f is diff in (a, b)

Then, $\exists\ c \in (a, b)$, s.t $\boxed{f'(c) = \frac{f(b)-f(a)}{(b-a)}}$

Geometrically,



Analytically, consider

$$F(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}\right) x$$

Observe, $F(a) = F(b) = \frac{b f(a) - a f(b)}{(b-a)}$

Applying Rolle's theorem on $F(x)$,

$\exists \; c \in (a,b)$ s.t $F'(c) = 0$

$$\Rightarrow f'(c) - \left(\frac{f(b)-f(a)}{b-a}\right) = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

□

→ Intermediate Value Theorem (IMVT)

If $f(x)$ is a cont. fn in $x \in [a,b]$, then it takes on any given value b/w $f(a)$ & $f(b)$ at some pt. within the interval.

Q 1. If $a+b+c=0$, then P.T $3ax^2+2bx+c=0$ has atleast one root in $(0,1)$

2. How many rts. of the eqn $(x-1)(x-2)(x-3) + (x-1)(x-2)(x-4) + (x-2)(x-3)(x-4) + (x-1)(x-3)(x-4) = 0$ are positive ?

3. P.T b/w any two rts of $e^{-x} - \cos x = 0$, there exists at least on rt of $\sin x - e^{-x} = 0$

4. Find 'c' of LMVT for $f(x) = \sqrt{25-x^2}$ in $[1,5]$

5. Using LMVT, show that

$$\frac{\beta-\alpha}{1+\beta^2} < t^{-1}\beta - t^{-1}\alpha < \frac{\beta-\alpha}{1+\alpha^2} ;$$

$0 < \alpha < \beta < \pi/2$

A. 1. $f(x) = ax^3 + bx^2 + cx + d$

$f(0) = d$ & $f(1) = \overbrace{a+b+c}^{0}+d = \underline{d}$

(RT)

$\Rightarrow \exists \ c \in (0,1)$; s.t $f'(c) = 0$

$\Rightarrow \underline{3ax^2 + 2bx + c = 0}$

2. $f(x) = (x-1)(x-2)(x-3)(x-4)$

$f(1) = f(2) = f(3) = f(4) = 0$

(RT)

$\Rightarrow \exists \ c_1 \in (1,2)$

$c_2 \in (2,3)$ s.t $\underline{f'(c) = 0}$

$c_3 \in (3,4)$

$\because f'(x)$ is a cubic polynomial $\Rightarrow$ Max 3 rts

$\Rightarrow$ All rts +ve

3. Let $f(x) = e^{-x} - cx = 0 \begin{matrix} x_1 \\ x_2 \end{matrix}$ $\Rightarrow$ $\underline{f(x_1) = f(x_2) = 0}$

(RT)

$\Rightarrow \exists \ c \in (x_1, x_2)$, $f'(c) = 0$

$\Rightarrow sc - e^{-c} = 0$

$\Rightarrow sx - e^{-x} = 0 \leftarrow \underline{c \in (x_1, x_2)}$

□

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4. $f(1) = \sqrt{24}$ $\xrightarrow{(LT)}$ $f'(c) = \dfrac{f(5)-f(1)}{(5-1)} = -\dfrac{\sqrt{24}}{4}$

$f(5) = 0$

$= -\dfrac{\sqrt{6}}{2}$; $c \in [1,5]$

$$f'(x) = \frac{-x}{\sqrt{25-x^2}} = -\frac{\sqrt{6}}{2}$$

$\Rightarrow$ $2x = \sqrt{6}\sqrt{25-x^2}$

$\Rightarrow$ $4x^2 = 150 - 6x^2 \Rightarrow$ $\underline{x = \sqrt{15}}$

5. $f(x) = \tan^{-1}x$ $\xrightarrow{(LT)}$ $f'(c) = \dfrac{f(\beta)-f(\alpha)}{\beta-\alpha}$

$f'(x) = \dfrac{1}{1+x^2}$

$= \dfrac{\tan^{-1}\beta - \tan^{-1}\alpha}{\beta-\alpha}$

$f''(x) = \dfrac{-2x}{(1+x^2)^2} < 0$ $\forall\, x \in (\alpha, \beta)$

$\Rightarrow f'(x) \downarrow$ $\because$ $\alpha < c < \beta$

$\Rightarrow f'(\beta) < f'(c) < f'(\alpha)$

$\Rightarrow \dfrac{1}{1+\beta^2} < \dfrac{\tan^{-1}\beta - \tan^{-1}\alpha}{\beta-\alpha} < \dfrac{1}{1+\alpha^2}$

$\Rightarrow \underline{\left(\dfrac{\beta-\alpha}{1+\beta^2}\right) < \tan^{-1}\beta - \tan^{-1}\alpha < \left(\dfrac{\beta-\alpha}{1+\alpha^2}\right)}$

Q1 If $f(x) = x^{\alpha}\ell(x)$ & $f(0)=0$, then find α for which rolle's theorem can be applied in $[0,1]$

Q2 If $f(x)$ & $g(x)$ are diff for $x \in [0,1]$ s.t $f(0)=2$, $g(0)=0$, $f(1)=6$, $g(1)=2$, then show that $\exists$ c satisfying $c \in (0,1)$ & $f'(c) = 2g'(c)$

Q3 If $f(x)$ is a twice diff fn^n and given that $f(1)=1$, $f(2)=4$, $f(3)=9$, then P.T $f''(x)=2$, for some $x \in (1,3)$

A1 For RT,
① $f(x)$ cont in $[0,1]$.
② $f(x)$ diff in $(0,1)$
③ $f(1)=f(0)=0$. ⟶ $\underline{\alpha \in \mathbb{R}}$

① $\underline{x=0}$ R: $\lim\limits_{h\to 0^+} h^{\alpha}\ell(h) = \dfrac{\ell(h)}{h^{-\alpha}}$

$$\overset{(LH)}{=} \frac{1/h}{-\alpha h^{-(\alpha+1)}} = \frac{-1}{\alpha}(h^{\alpha}) = 0$$

$\Rightarrow \boxed{\alpha > 0}$

2- $F(x) = f(x) - 2g(x)$
$F(0) = f(0) - 2g(0) = 2 - 2(0) = \underline{2}$
$F(1) = f(1) - 2g(1) = 6 - 2(2) = \underline{2}$

$F(0) = F(1) \overset{(RT)}{\Rightarrow} \exists\ c \in (0,1)$, $F'(c) = 0 \Rightarrow \underline{f'(c) = 2g'(c)}$

3. $F(x) = f(x) - x^2$

$F(1) = F(2) = F(3) = 0$

$\overset{(RT)}{\Rightarrow} \quad \exists \quad c_1 \in (1,2) \qquad F'(c_1) = 0$

$\qquad\qquad c_2 \in (2,3) \qquad F'(c_2) = 0$

$\overset{(RT)}{\Rightarrow} \quad \exists \quad c \in (c_1, c_2) \qquad F''(c) = 0 \Rightarrow \underline{f''(c) = 2}$

Q4. Using RT, PT $\exists$ rt $\in (45^{\frac{1}{100}}, 46)$ of the eqn

$$P(x) = 51x^{101} - 2323x^{100} - 45x + 1035 = 0$$

Q5. Find all the critical pts of

$$f(x) = \begin{cases} (2+x)^3, & x \in [-3,-1] \\ x^{2/3}, & x \in (-1,2) \end{cases}$$

Q6. Find no. of pts in $(-\infty, \infty)$ for which

$$x^2 - x\,sx - Cx = 0$$

A 4. $f(x) = \frac{x^{102}}{2} - 23x^{101} - \frac{45x^2}{2} + 1035x$

$f(46) = \frac{(46)(46)^{101}}{2} - 23x^{101} - \frac{45(46)^2}{2} + \overset{23\times45}{\overline{(1035)}}(46)$

$= \underline{0}$

$f(45^{\frac{1}{100}}) = \left(\frac{45}{2}\right)(45)^{\frac{2}{100}} - 23(45)(45)^{\frac{1}{100}} - \frac{45}{2}(45)^{\frac{2}{100}} + 1035(45)^{\frac{1}{10}}$

$= \underline{0}$

$f(45^{1/100}) = f(46) \overset{(RT)}{\Rightarrow} \exists\ c \in (45^{\frac{1}{100}}, 46)$

s.t $\underline{f'(c) = 0}$

5. $f'(x) = \begin{cases} 3(x+2)^2, & x \in (-3,-1] \\ \frac{2}{3x^{1/3}}, & x \in (-1,2) \end{cases}$

$\left.\begin{matrix} f'(-1^-) = 3 \\ f'(-1^+) = \frac{-2}{3} \end{matrix}\right\} \Rightarrow f'(-1)$ does $\underline{\text{not}}$ exist.

& $f'(0)$ $\underline{\text{not}}$ defined. $\Rightarrow$ $\underline{\{0, -1, -2\}}$

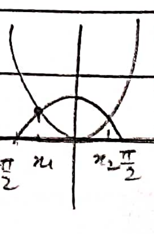
$f'(-2) = 0$

6. $f(x) = \frac{x^3}{3} + x c_x = x\left(\frac{x^2 - 3c_x}{3}\right)$

$f(0) = f(x_1) = f(x_2) = 0$

$\Rightarrow \exists\ c_1 \in (x_1, 0)$ $f'(c_1) = 0$ $\Rightarrow$ $\boxed{\text{2 values}}$

$c_2 \in (0, x_2)$ $f'(c_2) = 0$



Alternate Method $\qquad g(x) = x^2 - xSx - Cx$

$g(0) = -1 \qquad g'(x) = 2x - Sx - xCx + Sx$

$= x(2 - Cx)$

$\because \lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(x) = \infty$

$\Downarrow$

(2 values) for $\underline{g(x) = 0}$

$\Rightarrow$ Min at $\underline{x=0}$ $\qquad - \;\; + \;\; \overset{}{\underset{0}{|}}$

35 $(45)^{100}$

Q7. Let $P(x)$ be a polynomial of degree 4 with extrema at $x = 1, 2$

& $\lim_{x\to 0}\left(\dfrac{x^2 + P(x)}{x^2}\right) = 2$

Find $P(2)$.

Q8. Find the total # distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$.

A. 7. $P'(x) = k(x-1)(x-2)$

① Must be (0/0) for lt to exist $\rightarrow \boxed{P(0) = 0}$

$\lim_{x\to 0} \dfrac{x^2 + P(x)}{x^2} \underset{1\,(LH)}{\Rightarrow} \dfrac{2x + P'(x)}{2x} \underset{2\,(LH)}{=} \dfrac{P''(x)}{2} = 2$

② $\Rightarrow \boxed{P'(0) = 0} \Rightarrow \underline{P''(0) = 4}$

$\Downarrow$

$\underline{P'(x) = kx(x-1)(x-2)}$

$P''(0) = k[2] = 4 \Rightarrow \underline{k = 2}$

$\Rightarrow P'(x) = 2x^3 - 6x^2 + 4x$

$P''(x) = \dfrac{x^4}{2} - 2x^3 + 2x^2 + C$

$P''(0) = 0 \Rightarrow C = 0 \Rightarrow P(x) = \dfrac{x^4}{2} - 2x^3 + 2x^2$

$P(2) = 8 - 16 + 8 = \underline{0}$

8. $f'(x) = 4x^3 - 12x^2 + 24x + 1$

$f''(x) = 12x^2 - 24x + 24 = 12(x-1)^2 + 12$

$f''(x) > 0 \quad \forall\ x \in \mathbb{R} \Rightarrow f'(x) \uparrow$

$\Rightarrow \exists\ x_0 \quad f'(x_0) = 0 \quad \because f'(0) = 1 \Rightarrow \underline{x_0 < 0}$

$\forall\ x > x_0 \quad f(x) \uparrow \Rightarrow f(x) > f(x_0)$

$\forall\ x < x_0 \quad f(x) \downarrow \Rightarrow f(x) > f(x_0)$

$\Rightarrow f(x_0) < f(0) = -1 \Rightarrow \underline{f(x_0) < 0}$

$\Downarrow$

$\boxed{2 \text{ rts}}$

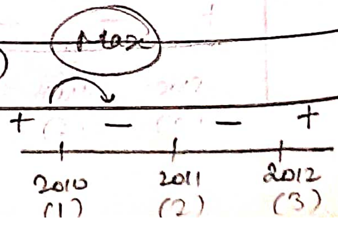
Q9. Let $f(x) = 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4$ $\forall\ x \in \mathbb{R}$.
If $g : \mathbb{R} \to (0, \infty)$, $f(x) = \ell(g(x))$ $\forall\ x \in \mathbb{R}$, then find the total # $x \in \mathbb{R}$ at which $g(x)$ has local max.

Q10. Find the total # local max/min of
$f(x) = |x| + |x^2 - 1|$

A9. $g(x) = e^{f(x)}$

$\Rightarrow g'(x) = e^{f(x)} f'(x)$

$f'(x) = 2010\,(x-2010)(x-2011)^2(x-2012)^3\,(\sim)$



$\Rightarrow \underline{1 \text{ Max}}$

10. $$f(x) = \begin{cases} x^2 - x + 1 & , x < -1 \\ -x^2 - x + 1 & , x \in [-1, 0) \\ -x^2 + x + 1 & , x \in [0, 1) \\ x^2 + x - 1 & , x \geq 1 \end{cases}$$

$$f'(x) = \begin{cases} 2x - 1 & , x \leq -1 \\ -2x - 1 & , x \in [-1, 0) \\ -2x + 1 & , x \in [0, 1) \\ 2x + 1 & , x \geq 1 \end{cases}$$

$f'(x) = 0$ at $\underline{x = -\frac{1}{2}, \frac{1}{2}}$ & does not exist at $0, -1, 1$.

$$f''(x) = \begin{cases} -2 & , x < 0 \\ -2 & , x \in [0, 1) \\ 2 & , x \geq 1 \end{cases} \Rightarrow f''(x) \neq 0$$

for any critical pt.

$\therefore$ 5 local extremum

14/06/2023

Q1. Let $P(x)$ be a real polynomial of least degree which has a local max at $x=1$ & local min at $x=3$. If $P(1)=6$, $P(3)=2$, find $P'(0)$.

Q2. A cubic fxn $f(x)$ vanishes at $x=-2$ & has min & max value at $x=-1$, $x=1/3$ respectively. If $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$, find $f(x)$.

Q3. Let $p \in [-1, 1]$. Show that the eqn $4x^3 - 3x - p = 0$ has a unique root in $x \in [\frac{1}{2}, 1]$

A1. $P'(x) = k(x-1)(x-3) = kx^2 - 4kx + 3k$

$$P(x) = \frac{k}{3}x^3 - 2kx^2 + 3kx + c$$

$$P(1) = \frac{4k}{3} + c = 6 \Rightarrow \underline{k=3}$$

$$P(3) = 9k - 18k + 9k + c = 2 \Rightarrow \underline{c=2}$$

$$P'(0) = 3k = \underline{\textcircled{9}}$$

2. $f'(x) = k(x+1)(3x-1) = 3kx^2+2kx-k$

$f(x) = kx^3+kx^2-kx+c$

$f(-2) = -8k+4k+2k+c=0 \Rightarrow \underline{-2k+c=0}$

$\int_{-1}^{1} f(x)\,dx = \left[\frac{kx^4}{4}+\frac{kx^3}{3}-\frac{kx^2}{2}+cx\right]_{-1}^{1}$

$= \underline{\frac{2k}{3}+2c = \frac{14}{3}} \Rightarrow \frac{k}{3}+c=\frac{7}{3}$

$\underline{k=1}$

$\underline{c=2}$

$f(x) = \underline{x^3+x^2-x+2}$

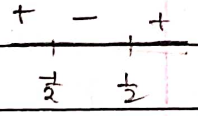
3. $f(1) = 1-p$ $\qquad p \in [-1,1] \Rightarrow 1-p \in [0,2]$

$f\left(\frac{1}{2}\right) = \frac{1}{2}-\frac{3}{2}-p$ $\qquad \Rightarrow -(1+p) \in [-2,0]$

$= -1-p$

$\Rightarrow f(1)\,f\left(\frac{1}{2}\right) \leq 0 \Rightarrow \exists\ c \in \left[\frac{1}{2},1\right]$ $\underline{f(c)=0}$

$f'(x) = 12x^2-3 = 0 \Rightarrow \underline{x=\pm\frac{1}{2}}$



$\Rightarrow f'(x) > 0\ \forall\ x \in \left[\frac{1}{2},1\right] \Rightarrow f(x)\uparrow \Rightarrow$ <u>unique rt</u>

<u>Q4.</u> Using the relⁿ $2(1-\cos x) < x^2,\ x \neq 0$, or otherwise, P.T $\sin(\tan x) \geq x,\ \forall\ x \in \left[0,\frac{\pi}{4}\right]$

*<u>Q5.</u> P.T for $x \in [0,\pi/2]$, $\sin x + 2x \geq \frac{3x(x+1)}{\pi}$

A 4. $f(x) = s(t_x) - x$

$$f'(x) = \frac{c_{t_x}}{c_x^2} - 1 = (c_{t_x})(1+t_x^2) - 1$$

$$> \left(1 - \frac{t_x^2}{2}\right)(1+t_x^2) - 1$$

$2 - 2c_{t_x} < t_x^2$ $\quad > \dfrac{t_x^2(1-t_x^2)}{2} \geq 0$

$\Rightarrow c_{t_x} > \left(\dfrac{2-t_x^2}{2}\right)$

$x \in [0, \frac{\pi}{4}] \Rightarrow t_x \in [0,1]$ $\quad \Rightarrow f'(x) > 0 \ \forall x \in [0, \frac{\pi}{4}]$

$\Rightarrow f(x) \uparrow$

$\Rightarrow f(x) \geq f(0) \Rightarrow \boxed{s(t_x) \geq x}$ □

☆ 5. $f(x) = s_x + 2x - \dfrac{3x(x+1)}{\pi}$

$f'(x) = c_x + 2 - \dfrac{3}{\pi}(2x+1)$

$f''(0) = -\left(s_x + \dfrac{6}{\pi}\right) \Rightarrow f'(x) \downarrow$

$f'(0) = 3 - \dfrac{3}{\pi}$ $\qquad f'(\frac{\pi}{2}) = -3 - \dfrac{3}{\pi}$

$\because f'(0) f'(\frac{\pi}{2}) < 0 \Rightarrow \exists x_0 \in [0, \frac{\pi}{2}]$ s.t. $f'(x_0) = 0$.

$\Rightarrow x \in [0, x_0)$, $f(x) \uparrow$

$x \in (x_0, \pi/2]$, $f(x) \downarrow$

$f(0) = 0 \Rightarrow \forall x \in [0, x_0)$, $\underline{f(x) > 0}$

$f(\frac{\pi}{2}) = 1 + \pi - \frac{3}{2}(\frac{\pi}{2} + 1)$

$= \dfrac{\pi}{4} - \dfrac{1}{2} > 0 \Rightarrow \forall x \in (x_0, \pi/2]$, $\underline{f(x) > 0}$

Q6. If $P(x)$ is a polynomial of degree 3 satisfying $P(-1) = 10$, $P(1) = -6$ & $P(x)$ has max at $x = -1$ & $P'(x)$ has min at $x = 1$, find the dist. b/w the pt. of local max & min of curve.

☆ Q7. If $P(1) = 0$ & $\frac{dP(x)}{dx} > P(x)$ $\forall\ x \geq 1$, then P.T $P'(x) > 0$ $\forall\ x > 1$

Q8. If $f(x) = x^3 + e^{x/2}$ & $g(x) = f^{-1}(x)$, then find the value of $g'(1)$.

Q9. Let $f : \mathbb{R} \longrightarrow (-1, 1)$ s.t. $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$ $\forall\ x \in (-1, 1)$ and let f^{-1} be the inverse of f. Find $(f^{-1})'(2)$.

A6. $P''(x) = kx - k$

$P'(x) = \frac{kx^2}{2} - kx + c$

$$\begin{array}{c} + \quad - \\ \hline 1 \end{array}$$

$\Rightarrow \underline{k < 0}$

$P'(-1) = 0 \Rightarrow \frac{3k}{2} + c = 0$

$P(x) = \frac{kx^3}{6} - \frac{kx^2}{2} + cx + d$

$P(1) = -\frac{k}{3} + c + d = -6$

$P(-1) = -\frac{2k}{3} - c + d = 10$

$\frac{11k}{6} - d = 6$

$\frac{5k}{6} + d = 10$

$\Rightarrow \frac{16k}{6} = 16 \Rightarrow \underline{k = 6}$

$\underline{d = 5}$

$\underline{c = -9}$

$P(x) = x^3 - 3x^2 - 9x + 5$

3. $g(f(x)) = x \Rightarrow g'(f(x))\, f'(x) = 1$

$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$

$f(0) = 1 \Rightarrow g'(1) = \frac{1}{f'(0)} = \textcircled{2}$

$f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$

4. $g(x) = f^{-1}(x) \Rightarrow g(f(x)) = x \Rightarrow g'(f(x))\, f'(x) = 1$

$\Rightarrow g'(f(0))\, f'(0) = 1$

$f(0) = 2$

$\Rightarrow \underline{g'(2) = \frac{1}{3}}$

$e^{-x}(f'(x) - f(x)) = \sqrt{x^4+1}$

$\Rightarrow f'(0) - f(0) = 1$

$\Rightarrow \underline{f'(0) = 3}$

★ 2. $\frac{dP(x)}{dx} - P(x) > 0 \Rightarrow e^{-x}\left(\frac{dP(x)}{dx}\right) - e^{-x}P(x) > 0$

$\Rightarrow \frac{d}{dx}\left(e^{-x}P(x)\right) > 0$

$\Rightarrow e^{-x}P(x) \uparrow \quad \forall\ x > 1$

$\Rightarrow e^{-x}P(x) > e^{-1}P(1)$

$$\left.\begin{array}{l} \Rightarrow \underline{P(x) > 0} \\ \\ \text{Since } \frac{dP(x)}{dx} > P(x) \end{array}\right\} \Rightarrow \underline{P'(x) > 0}$$

Q10. Let $f(x) = |ax-b| + c|x|$, $\forall x \in \mathbb{R}$, $a, b, c > 0$. Find condn on a, b, c if $f(x)$ attains min value at exactly one pt.

Q11. $f : [2, 7] \to [0, \infty)$. cont. & diff.
Then show that

$$(f(7)-f(2)) \left[\frac{(f(7))^2 + f(7)f(2) + f^2(2)}{3} \right] = 5f^2(c)f'(c),$$
$c \in (2, 7)$

Q12. Let a, b, c be non-zero real nos. st.

$$\int_0^1 (1+c_x^8)(ax^2+bx+c)\,dx = \int_0^2 (1+c_x^8)(ax^2+bx+c)\,dx = 0$$

Show that the eqn $ax^2+bx+c=0$ has one rt. b/w 0 & 1 and other rt. b/w 1 & 2

A10.
$$f(x) = \begin{cases} b-(a+c)x, & x < 0 \\ b+(c-a)x, & x \in [0, \frac{b}{a}) \\ (a+c)x-b, & x \geq b/a \end{cases}$$

For min at exactly one pt, $f(0) \neq f(\frac{b}{a})$

$\Rightarrow b \neq \frac{bc}{a}$

$\Rightarrow \boxed{a \neq c}$

11. LMVT on $(f(x))^3$

$$\Rightarrow \quad 3f^2(c)\,f'(c) = \frac{f^3(7)-f^3(2)}{5}$$

$$\Rightarrow \quad 5f^2(c)\,f'(c) = \boxed{\frac{f^3(7)-f^3(2)}{3}}$$

12. $\exists F(x) = \int_0^x (1+c^8_x)(ax^2+bx+c)$

$$F(0) = F(1) = F(2) = 0 \quad \xrightarrow{(RT)} \quad \exists \; c_1 \in (0;1), \; F'(c_1) = 0$$

$$c_2 \in (1,2), \; F'(c_2) = 0$$

$$F'(x) = (\underbrace{1+c^8_x}_{\neq 0 \; \forall x \in \mathbb{R}})(ax^2+bx+c) = 0$$

$$\Rightarrow \quad \underline{ax^2+bx+c=0} \begin{matrix} \nearrow c_1 \\ \searrow c_2 \end{matrix}$$

☆ Q13. For a twice diff. f^n $f(x)$, $g(x)$ is defined as

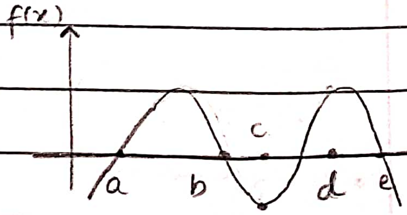
$$g(x) = (f'(x))^2 + f''(x)\,f(x) \quad \text{on } [a,e]$$

If $a < b < c < d < e$ &

$f(a) = 0$, $f(b) = 2$, $f(c) = -1$, $f(d) = 2$, $f(e) = 0$,

then find min no. of zeroes of $g(x)$.

A13. $$g(x) = \frac{d}{dx}\left(\underbrace{f(x)}_{\downarrow \; 4 \text{ rts}}\;\underbrace{f'(x)}_{\downarrow \; 3 \text{ rts}}\right)$$



$\Rightarrow$ $g(x)$ has $\underline{\text{min } 6 \text{ rts}}$

Q14. If the fxn $f: [0,4] \to \mathbb{R}$ is a diff fxn, then show that for $a, b \in (0,4)$.

(i) $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$

(ii) $\int_0^4 f(t)\, dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)]$ $\forall\ 0<\alpha,\beta<2$

A.14 (i) $2f(a)f'(a) = \frac{f(4)^2 - f(0)^2}{4}$ (LMVT)

$\Rightarrow f'(4) - f^2(0) = 8 f'(a) f(a)$

We can choose $\underline{b=a}$

But if in Q, $\underline{a \neq b}$ given

(LMVT) $f'(a) = \frac{f(4) - f(0)}{4}$

(IMVT) $f(b) = \frac{f(4) + f(0)}{2}$

$\Big\}\ f^2(4) - f^2(0) = 8 f'(a) f(b)$

(ii) Let $f(x) = \int_0^{x^2} f(t)\, dt = \int_0^x f(u^2)(2u)\, du$

$t = u^2$

$\Rightarrow dt = 2u\, du$

$f'(x) = 2x f(x^2)$

(LMVT) $f'(\alpha) = \frac{f(2) - f(0)}{2} = \frac{\int_0^4 f(x)\, dt}{2}$; $\alpha \in (0,2)$

$\Rightarrow \int_0^4 f(t)\, dt = 2[2\alpha f(\alpha^2)]$

We can choose $\beta = \alpha$ $\Rightarrow \int_0^4 f(t)\, dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)]$